

# ON THE POSTPROJECTIVE PARTITIONS AND COMPONENTS OF THE AUSLANDER-REITEN QUIVERS

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**ABSTRACT.** In this paper we shall investigate further the connections between the postprojective partition of an algebra and its Auslander-Reiten quiver.

Auslander-Smalø introduced, in [3], the notion of postprojective partition and modules (under the name of preprojective). The connection between such a partition and the structure of the Auslander-Reiten quiver has been investigated in several papers such as [1, 3, 6, 7, 8, 12]. The purpose of this paper is to follow such investigations.

We introduce the notion of  $\mathbf{P}$ -discrete component of the Auslander-Reiten quiver  $\Gamma_A$  as follows. Let  $\{\mathbf{P}_i\}$  of  $\text{ind}A$  with  $i \in \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$  be the postprojective partition of  $A$  (recall the definition below). A component  $\Gamma$  of  $\Gamma_A$  is  $\mathbf{P}$ -discrete if for each  $i \geq 0$  and each  $M \in \Gamma \cap \mathbf{P}_i$ , we have that  $\text{tr}_{\mathbf{P}_{i+1}}(M) = \text{tr}_{\mathbf{P}_\infty}(M)$ , where  $\text{tr}_{\mathcal{C}}(M)$  denotes the trace of the set of modules  $\mathcal{C}$  in  $M$  (see Section 1 below).

**Theorem 2.3.** *Let  $A$  be a representation-infinite Artin algebra. If  $\Gamma$  is a  $\mathbf{P}$ -discrete connected component of  $\Gamma_A$  then there is no arrow  $M \rightarrow N$  in  $\Gamma$  with  $M \in \mathbf{P}_i$  and  $N \in \mathbf{P}_j$  such that  $i + 1 < j < \infty$ .*

Also, using the notion of left degree of a morphism (introduced by Liu in [10]), we prove the following result.

**Theorem 3.4.** *Let  $A$  be a finite dimensional algebra over an algebraically closed field and let  $f: M \rightarrow N$  be an irreducible monomorphism of infinite left degree. If  $\text{tr}_{\mathbf{P}_\infty}(N) = 0$ , then  $\text{Coker} f \in \mathbf{P}_\infty$  and every non-trivial submodule of  $\text{Coker} f$  is postprojective.*

In particular, from the last theorem we get that if an irreducible monomorphism which lies in a postprojective component has its cokernel in a regular component then the latter must be simple regular.

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This paper is organized as follows. After recalling basic notions in Section 1, we prove Theorem 2.3 in Section 2 and Theorem 3.4 in Section 3.

## 1. PRELIMINARIES

**1.1. Basics.** For the results of Section 2, we assume the algebras  $A$  to be Artin algebras, unless otherwise stated. For the last section, we shall restrict to finite dimensional algebras over a fixed algebraically closed field  $k$ . Furthermore, we will assume that all algebras are basic.

For unexplained notions in representation theory we refer the reader to [4].

For an algebra  $A$ , we denote by  $\text{mod}A$  the category of all finitely generated left  $A$ -modules, and by  $\text{ind}A$  the full subcategory of  $\text{mod}A$  consisting of one representative of each isomorphism class of indecomposable  $A$ -modules. We denote by  $\Gamma_A$  the Auslander-Reiten quiver of  $A$  and by  $\tau$  and  $\tau^-$  the Auslander-Reiten translations  $\text{DTr}$  and  $\text{TrD}$ , respectively.

Given  $n \geq 1$  and  $M, N \in \text{mod}A$ , we define the subgroups  $\text{rad}^n(M, N)$  of  $\text{Hom}(M, N)$  by induction: for  $n = 1$ , we set  $\text{rad}^1(M, N)$  to be the set of all morphisms  $f: M \rightarrow N$  such that the compositions  $ghf$  are not isomorphisms for all  $h: L \rightarrow M$  and  $g: N \rightarrow L$ , with  $L$  indecomposable. Also, we define  $\text{rad}^n(M, N)$  as the set of all morphisms  $f \in \text{Hom}(M, N)$  such that there exist  $X \in \text{mod}A$  and morphisms  $g \in \text{rad}(M, X)$  and  $h \in \text{rad}^{n-1}(X, N)$  such that  $f = hg$ . Finally, we set  $\text{rad}^\infty(M, N) = \bigcap_{n \geq 1} \text{rad}^n(M, N)$ .

We recall that for  $X, Y \in \text{ind}A$ ,  $f: X \rightarrow Y$  is called **irreducible** if and only if  $f \in \text{rad}(X, Y) \setminus \text{rad}^2(X, Y)$ . A **path of irreducible morphisms of length  $n$**  is a sequence  $M_0 \xrightarrow{h_1} M_1 \rightarrow \cdots \rightarrow M_{n-1} \xrightarrow{h_n} M_n$  where each  $h_i$  is irreducible and each  $M_j$  is indecomposable.

Following Liu [10], we say that the left degree of an irreducible morphism  $f: X \rightarrow Y$  is  $n$ , and we denote  $d_l(f) = n$ , if  $n$  is the smallest positive integer for which there exist  $Z \in \text{ind}A$  and a morphism  $h: Z \rightarrow X$  such that  $h \in \text{rad}^n(Z, X) \setminus \text{rad}^{n+1}(Z, X)$  and  $fh \in \text{rad}^{n+2}(Z, Y)$ . In case this condition is not verified for any  $n \geq 1$  we say the left degree of  $f$  is infinite. Dually, one can define the right degree of an irreducible morphism.

**1.2. Postprojective partitions and modules.** We shall now recall the concept of postprojective partition and modules as introduced by Auslander and Smalø in [3] under the name preprojective.

A **postprojective partition** of an Artin algebra  $A$  is a partition  $\{\mathbf{P}_i\}$  of  $\text{ind}A$  with  $i \in \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$  such that

- (a)  $\text{ind}A$  is the disjoint union of the subcategories  $\mathbf{P}_i$ ,  $i \in \mathbb{N}_\infty$ .
- (b) for each  $j < \infty$ ,  $\mathbf{P}_j$  is a finite minimal cover of the union of the subcategories  $\mathbf{P}_i$  such that  $j \leq i \leq \infty$ .

It is clear that the  $A$ -modules in  $\mathbf{P}_0$  are all the indecomposable projectives. In this article, we denote  $\mathbf{P}(\text{ind}A) = \bigcup_{0 \leq i < \infty} \mathbf{P}_i$  simply by  $\mathbf{P}$ .

The modules in  $\text{add}\mathbf{P}$  will be called **postprojective** modules (former preprojective modules in [3]). We denote by  $\mathbf{P}^m$  the subcategory  $\mathbf{P}_0 \cup \dots \cup \mathbf{P}_m$ .

Given  $i \in \mathbb{N}_\infty$  and a module  $M$  in  $\text{mod}A$  we denote the trace of  $\mathbf{P}_i$  on  $M$  by  $\text{tr}_{\mathbf{P}_i}(M)$ , that is, the submodule of  $M$  generated by the images of all morphisms which have domain in  $\text{add}\mathbf{P}_i$ . Therefore,  $\text{tr}_{\mathbf{P}_i}(M)$  is the submodule of  $M$  generated by  $\{\text{Im}f \mid f \in \text{Hom}(N, M) \text{ and } N \in \mathbf{P}_i\}$ . It was proved in [3] that  $\text{tr}_{\mathbf{P}_\infty}(M) = \bigcap_{i \geq 0} \text{tr}_{\mathbf{P}_i}(M)$ . Hence,  $\text{tr}_{\mathbf{P}_\infty}(M) = \text{tr}_{\mathbf{P}_r}(M)$  for some  $r \in \mathbb{N}$  since  $M$  is artinian and  $\text{tr}_{\mathbf{P}_{n+1}}(M) \subseteq \text{tr}_{\mathbf{P}_n}(M)$ , for each  $n \geq 0$ . It was also proved in [3] that  $M$  is postprojective if and only if  $\text{tr}_{\mathbf{P}_\infty}(M) \neq M$ .

The following proposition from [6] shall be very useful in the sequel.

**Proposition 1.1.** [6] *Let  $N$  be a postprojective module and  $f : M \rightarrow N$  a morphism such that  $\text{Im}f \not\subseteq \text{tr}_{\mathbf{P}_\infty}(N)$ . Then  $f \notin \text{rad}^\infty(M, N)$ .*

We also recall the following result from [8] (Lemma 4.2).

**Lemma 1.2.** *Let  $\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_\infty$  be the postprojective partition of an algebra  $A$ . Given  $0 < i \leq \infty$ , we have  $\text{Hom}(M, N) = \text{rad}^i(M, N)$ , for each  $M \in \mathbf{P}_0$  and each  $N \in \mathbf{P}_i$ .*

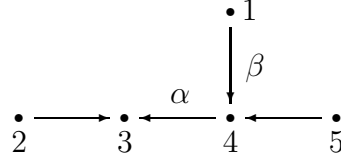
## 2. PATH OF IRREDUCIBLE MORPHISMS AND THE POSTPROJECTIVE PARTITION

Along this section, let  $A$  denote an Artin algebra and  $\{\mathbf{P}_i\}$  of  $\text{ind}A$  with  $i \in \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$  the postprojective partition of  $\text{ind}A$ . Let  $M$  be a postprojective module. It was proved in [3] that there exists a path of irreducible morphisms from a projective module  $P$  to  $M$ . Clearly, then,  $P$  and  $M$  lie in the same connected component  $\Gamma$  of  $\Gamma_A$ . One could wonder if there exists such a path as follows:

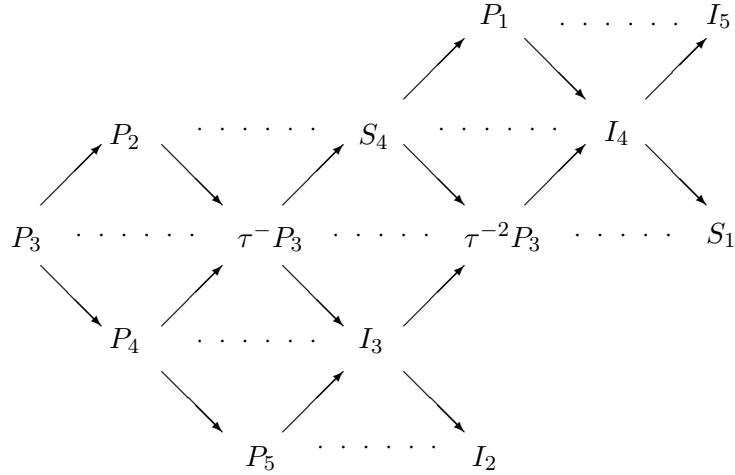
$$P = M_0 \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_n = M$$

with  $M_i \in \mathbf{P}_i$  for each  $i$ . Corollary 3 in [9] states that this is true if  $\Gamma$  is a postprojective component of a hereditary algebra. The next example show that this is not true in general. , and, on the other hand, that there are non-postprojective components with this property.

**Example 2.1.** Let  $A$  be the finite-dimensional  $k$ -algebra (where  $k$  is a field) given by the quiver



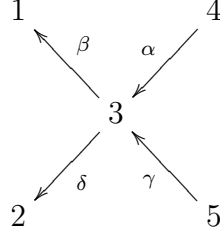
bound by  $\beta\alpha = 0$ . Its Auslander-Reiten quiver has the following shape:



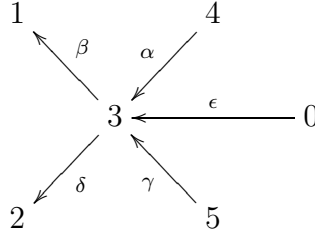
For each  $j$ ,  $P_j, I_j$  and  $S_j$  denote, respectively, the projective, the injective and the simple modules associated to the vertex  $j$  of the quiver. The postprojective partition is then  $\mathbf{P}_0 = \{P_1, P_2, P_3, P_4, P_5\}$ ,  $\mathbf{P}_1 = \{\tau^{-1}P_3, I_3, I_4\}$ ,  $\mathbf{P}_2 = \{S_4, \tau^{-2}P_3, I_1, I_2\}$  and  $\mathbf{P}_3 = \{I_5\}$ . Observe that there are paths from a projective to  $I_5 \in \mathbf{P}_3$  of length 2, 4, 5 and 6, and so, none of the required type.

Next example shows that there are non-postprojective components with this property.

**Example 2.2.** Let  $A$  be a path algebra defined by the quiver:

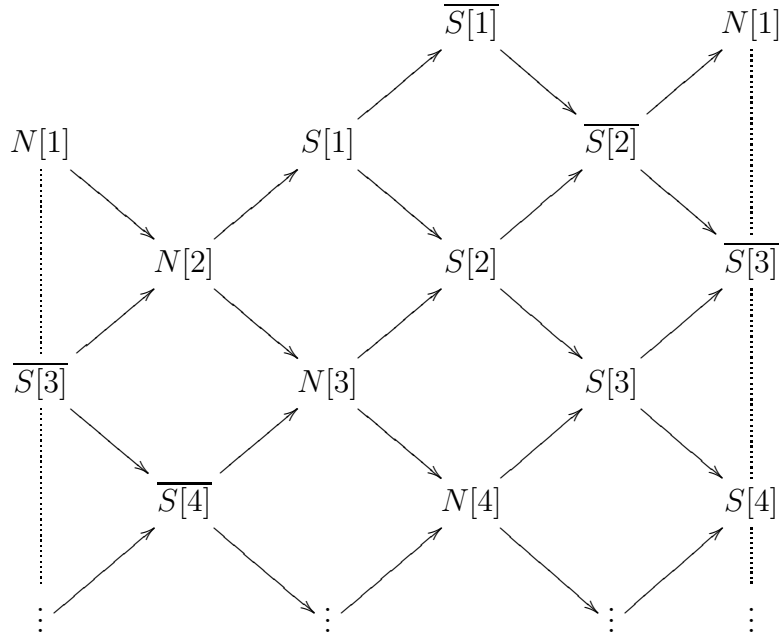


(see [8]). Let  $S$  be the simple module associated to the vertex 3 and let  $N$  be the indecomposable module such that  $\tau S = N$  and  $\tau N = S$ . One can check that  $S$  and  $N$  determine a tube of rank 2. The bounded quiver of the extended algebra  $A[S]$  is



bounded by  $\epsilon\beta = 0$  and  $\epsilon\delta = 0$ .

The ray tube which contains  $S = S[1]$  has the shape:



Let  $\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_\infty$  be the postprojective partition of  $A[S]$ . Then the modules  $N[i]$  and  $S[j]$  with  $i \geq 1$  and  $j \geq 1$  are all in  $\mathbf{P}_\infty$  and the ray  $\overline{S[1]} \rightarrow \overline{S[2]} \rightarrow \dots \rightarrow \overline{S[n]} \rightarrow \dots$  is such that  $\overline{S[n]} \in \mathbf{P}_{n-1}$  for each  $n \geq 1$ .

Our purpose in this section is to introduce the concept of  $\mathbf{P}$ -discrete components of  $\Gamma_A$  and show that, for them, we have an affirmative answer to the above question. We start with a lemma.

**Lemma 2.1.** *Let  $n$  be an integer greater than 0, and let  $M \in \mathbf{P}_n$ . For each  $j$ ,  $0 \leq j < n$ , there exists a path of irreducible morphisms  $L \rightsquigarrow M$  through postprojective modules with  $L \in \mathbf{P}_j$  and with composition not lying in  $\text{rad}^\infty(L, M)$ .*

*Proof.* Since  $M \in \mathbf{P}_n$  and  $j < n$ , by definition, there exists an epimorphism  $\bigoplus_{i=1}^r L_i \xrightarrow{g} M$ , with  $L_i \in \mathbf{P}_j$  for each  $i$ . Write  $g = [g_1, \dots, g_r]$ . Since  $M$  is postprojective,  $\text{tr}_{\mathbf{P}_\infty}(M) \neq M$  and so  $\text{Im } g \not\subseteq \text{tr}_{\mathbf{P}_\infty}(M)$ . Hence, there exists an  $l$  such that  $\text{Im } g_l \not\subseteq \text{tr}_{\mathbf{P}_\infty}(M)$ . Because of Proposition 1.1,  $g_l \notin \text{rad}^\infty(M_l, M)$ . By [4] (Proposition 7.4), we get the expression  $g_l = \sum_i \alpha_i + \beta$  where each  $\alpha_i$  is a path of irreducible morphisms with composite not lying in  $\text{rad}^\infty(L_l, M)$  and  $\beta \in \text{rad}^\infty(L_l, M)$ . Using Proposition 1.1 again, we infer that  $\text{Im } \beta \subseteq \text{tr}_{\mathbf{P}_\infty}(M)$ . Hence, for some  $i$ , the composition  $h$  of the path  $\alpha_i$  has image  $\text{Im } h \not\subseteq \text{tr}_{\mathbf{P}_\infty}(M)$  and so  $h \notin \text{rad}^\infty(L_l, M)$ . It remains to show that the path  $\alpha_i$  pass through only postprojective modules. Suppose  $\alpha_i$  is a path

$$L_l \overset{(*)}{\rightsquigarrow} N \overset{(**)}{\rightsquigarrow} M$$

where  $N \in \mathbf{P}_\infty$  and write by  $\gamma$  and  $\gamma'$  the (nonzero) compositions of the paths  $(*)$  and  $(**)$ , respectively. Hence  $\gamma'\gamma = h$ . Now,  $\text{Im } \gamma' \subseteq \text{tr}_{\mathbf{P}_\infty}(M)$  because  $N \in \mathbf{P}_\infty$  and so  $\text{Im } h \subseteq \text{Im } \gamma' \subseteq \text{tr}_{\mathbf{P}_\infty}(M)$ , a contradiction. This proves the lemma.  $\square$

**Definition 2.1.** Suppose  $A$  is representation-infinite. We say that a connected component  $\Gamma$  of  $\Gamma_A$  is a  **$\mathbf{P}$ -discrete component** if for all  $i \geq 0$  and for each postprojective module  $M$  in  $\Gamma$  with  $M \in \mathbf{P}_i$  we have  $\text{tr}_{\mathbf{P}_{i+1}}(M) = \text{tr}_{\mathbf{P}_\infty}(M)$ .

**Remark:** Note that  $\text{tr}_{\mathbf{P}_{i+1}}(M) = \text{tr}_{\mathbf{P}_\infty}(M)$  implies  $\text{tr}_{\mathbf{P}_j}(M) = \text{tr}_{\mathbf{P}_\infty}(M)$ , for each  $j > i$ .

**Proposition 2.2.** *Let  $A$  be a representation-infinite Artin algebra and  $\Gamma$  be a connected component of  $\Gamma_A$ . The following are equivalent:*

- (a)  $\Gamma$  is a  $\mathbf{P}$ -discrete component.

- (b) *There exists no arrow  $M \rightarrow N$  in  $\Gamma$  with  $M \in \mathbf{P}_j$ ,  $N \in \mathbf{P}_i$  and  $i < j < \infty$ .*
- (c) *There exists no path of irreducible morphisms  $M \rightsquigarrow N$  in  $\Gamma$  through indecomposable postprojective modules with  $M \in \mathbf{P}_j$ ,  $N \in \mathbf{P}_i$  and  $i < j < \infty$ .*

*Proof.* As one can easily see that (b) and (c) are equivalent we show (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Suppose there exists an irreducible morphism  $f : M \rightarrow N$  in  $\Gamma$  with  $M \in \mathbf{P}_j$ ,  $N \in \mathbf{P}_i$  and  $i < j < \infty$ . Assuming  $\Gamma$  is  $\mathbf{P}$ -discrete we have  $\text{Im} f \subseteq \text{tr}_{\mathbf{P}_j}(N) = \text{tr}_{\mathbf{P}_\infty}(N)$ . Then we can factorize  $f$  as follows:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow f' & \nearrow \hookrightarrow \\ & \text{tr}_{\mathbf{P}_\infty}(N) & \end{array}$$

From the fact that  $f$  is irreducible and  $N \neq \text{tr}_{\mathbf{P}_\infty}(N)$ , we get that  $f' : M \rightarrow \text{tr}_{\mathbf{P}_\infty}(N)$  is a split monomorphism and  $M$  is a summand of  $\text{tr}_{\mathbf{P}_\infty}(N)$ . This is an absurd since  $\text{tr}_{\mathbf{P}_\infty}(N) \in \text{add} \mathbf{P}_\infty$ .

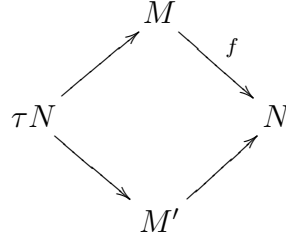
(c) $\Rightarrow$ (a) Suppose by contradiction that there exists  $M \in \Gamma \cap \mathbf{P}_i$ , with  $\text{tr}_{\mathbf{P}_{i+1}}(M) \neq \text{tr}_{\mathbf{P}_\infty}(M)$ . Then there exists  $f : M' \rightarrow M$ ,  $M' \in \mathbf{P}_{i+1}$ , such that  $\text{Im} f \not\subseteq \text{tr}_{\mathbf{P}_\infty}(M)$ . Therefore, Lemma 2.1 provides us a path of irreducible morphisms through indecomposable postprojective modules starting at  $M'$  and ending at  $M$  which contradicts (c).  $\square$

**Theorem 2.3.** *Let  $A$  be a representation-infinite Artin algebra. If  $\Gamma$  is a  $\mathbf{P}$ -discrete connected component of  $\Gamma_A$  then there is no arrow  $M \rightarrow N$  in  $\Gamma$  with  $M \in \mathbf{P}_i$  and  $N \in \mathbf{P}_j$  such that  $i + 1 < j < \infty$ .*

*Proof.* We shall prove it by induction on  $i \geq 0$ .

For  $i = 0$ , just observe that if  $f : P \rightarrow N$  is an irreducible morphism in  $\Gamma$  with  $P \in \mathbf{P}_0$  and  $N \in \mathbf{P}_j$ ,  $j > 1$ , then by Lemma 1.2 we have  $f \in \text{rad}^2(P, N)$  which is a contradiction.

Suppose now that the theorem is true for all values less than  $i$  and let  $f : M \rightarrow N$  be an irreducible morphism in  $\Gamma$  with  $M \in \mathbf{P}_i$ ,  $N \in \mathbf{P}_j$  and  $i + 1 < j < \infty$ . Then  $\tau N \in \mathbf{P}^{i-1}$  (Lemma 2.1 in [6]) and  $f$  is not a sink morphism because otherwise, by Lemma 2.1, there would be a path of irreducible morphisms through indecomposable postprojective modules  $L \rightarrow \cdots \rightarrow M \rightarrow N$ , with  $L \in \mathbf{P}_{i+1}$  which contradicts the fact that  $\Gamma$  is  $\mathbf{P}$ -discrete. Therefore, there exists  $M' \in \text{mod} A$  such that



is the Auslander-Reiten sequence which ends at  $N$ .

We now prove that  $M' \notin \text{add } \mathbf{P}^{j-2}$ . Suppose, by contradiction, that  $M' \in \text{add } \mathbf{P}^{j-2}$  and hence  $M \oplus M' \in \text{add } \mathbf{P}^{j-2}$ . Then there exists  $h : N' \rightarrow N$ ,  $N' \in \mathbf{P}_{j-1}$ , such that  $\text{Im } h \not\subseteq \text{tr}_{\mathbf{P}_\infty}(N)$ ,  $h \in \text{rad}(N', N)$ . Since  $h$  is not a split epimorphism, it can be lifted through  $f$ :

$$\begin{array}{ccccc}
& & N' & & \\
& g \swarrow & \downarrow h & & \\
M \oplus M' & \xrightarrow{f} & N & \longrightarrow & 0
\end{array}$$

Since  $\Gamma$  is  $\mathbf{P}$ -discrete, we know that  $\text{tr}_{\mathbf{P}_{j-1}}(M \oplus M') = \text{tr}_{\mathbf{P}_\infty}(M \oplus M')$  which implies that  $\text{Im } g \subseteq \text{tr}_{\mathbf{P}_\infty}(M \oplus M')$ . Therefore  $\text{Im } fg \subseteq \text{tr}_{\mathbf{P}_\infty}(N)$  since  $f(\text{tr}_{\mathbf{P}_\infty}(M \oplus M')) \subseteq \text{tr}_{\mathbf{P}_\infty}(N)$ . Hence  $\text{Im } h = \text{Im } fg \subseteq \text{tr}_{\mathbf{P}_\infty}(N)$ , which is a contradiction.

Then there exist a summand  $M_1 \in \mathbf{P}_{j-1}$  of  $M'$  and an irreducible morphism  $\tau N \rightarrow M_1$  such that  $\tau N \in \mathbf{P}^{i-1}$  and  $i-1 < i < j-1$  which contradicts the induction hypothesis. The theorem follows.  $\square$

**Remark:** In [12]( item (a) of Proposition 1), it was proved that if  $A$  is a hereditary algebra then there is no arrow  $M \rightarrow N$  in the postprojective component with  $M \in \mathbf{P}_i$  and  $N \in \mathbf{P}_j$  such that  $i+1 < j < \infty$ . Then this result was used to prove that the postprojective component satisfies item (b) of Proposition 2.2 (in other words, the converse of Theorem 2.3 for  $A$  hereditary).

**Corollary 2.4.** *Let  $A$  be a representation-infinite Artin algebra and  $\Gamma$  be a  $\mathbf{P}$ -discrete component of  $\Gamma_A$ . Given  $i > 0$  and  $M \in \mathbf{P}_i \cap \Gamma$ , then there exists a path of irreducible morphisms between indecomposable modules  $M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_i = M$ , where  $M_j \in \mathbf{P}_j$  for each  $j \in \{0, \dots, i\}$ , and moreover it has the smallest possible length among all the paths of irreducible morphisms starting at a projective and ending at  $M$ .*

*Proof.* Consider the sink map ending at  $M$ . Since  $M$  is not projective, this morphism is an epimorphism. Then there exists an irreducible



morphism  $N \longrightarrow M$  with  $N \in \mathbf{P}^{i-1}$ . In fact, by Theorem 2.3, we do have that  $N \in \mathbf{P}_{i-1}$ . We keep using the same argument until we get a path of irreducible morphisms  $M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_i = M$ , where  $M_j \in \mathbf{P}_j$  for each  $j \in \{0, \dots, i\}$ . Again by Theorem 2.3 we get that there can not be a smallest path starting at a projective and ending at  $M$ .  $\square$

### 3. RIGHT DEGREES OF IRREDUCIBLE MORPHISMS IN $\pi$ -COMPONENTS

Recall that if an irreducible morphism  $f : M \longrightarrow N$  in  $\text{mod} A$  has finite right degree then  $f$  is a monomorphism and we have  $d_r(f) = n$  if, and only if,  $\text{coker}(f) \in \text{rad}^n \setminus \text{rad}^{n+1}$  (by the dual version of Corollary 3.3 in [5]). We are now particularly interested in a particular case, assuming in addition that  $\text{tr}_{\mathbf{P}_\infty}(N) = 0$ . We look for a connection between the fact that the  $A$ -module  $\text{Coker} f$  is postprojective to the fact that the right degree of  $f$  is finite. Recall that, in [8], we have proved that the inclusion map  $f_S : \text{rad} P_S \hookrightarrow P_S$ , where  $P_S$  is the projective covering of the simple  $S$ , has finite right degree if and only if  $S$  is postprojective. From now on, since we depend on the results of [5], we shall restrict our consideration to finite dimensional algebras over an algebraically closed field  $k$ . Unless otherwise stated,  $A$  is such an algebra.

**Theorem 3.1.** *Let  $f : M \rightarrow N$  be an irreducible monomorphism with  $\text{tr}_{\mathbf{P}_\infty}(N) = 0$ . Then  $d_r(f) < \infty$  if and only if  $\text{Coker} f$  is postprojective.*

*Proof.* Suppose  $\text{Coker} f$  is postprojective. Then by the fact that  $\text{coker}(f)$  is an epimorphism and  $\text{tr}_{\mathbf{P}_\infty}(\text{Coker} f) \neq \text{Coker} f$  we have  $\text{coker}(f) \notin \text{rad}^\infty(N, \text{Coker} f)$ , by Proposition 1.1, so we get  $d_r(f) < \infty$ .

Now assume  $d_r(f) = n$ ,  $1 \leq n < \infty$ , and suppose by contradiction  $C = \text{Coker} f \in \mathbf{P}_\infty$ .

We set  $\pi_f = \text{coker}(f)$ . Then  $\pi_f \in \text{rad}^n(N, C) \setminus \text{rad}^{n+1}(N, C)$ , by Proposition 3.5 in [5]. We know there exists  $1 \leq j < \infty$  such that  $\text{tr}_{\mathbf{P}_j}(N) = \text{tr}_{\mathbf{P}_\infty}(N) = 0$ . Moreover, there exists a nonzero morphism  $v : L \rightarrow C$  with  $L \in \text{add} \mathbf{P}_j$  such that  $v \in \text{rad}^{n+1}(L, C)$ . Indeed, if we take  $r > j+n$  then we can get a covering  $h_r : M_r \rightarrow C$  with  $h_r \in \text{rad}(M_r, C)$  and  $M_r \in \text{add} \mathbf{P}_r$ , since  $C \in \mathbf{P}_\infty$ . Then let  $h_l : M_l \rightarrow M_{l+1}$  be a covering with  $M_l \in \text{add} \mathbf{P}_1$  for all  $l \in \{j, \dots, r-1\}$ . If we set  $v = h_r h_{r-1} \cdots h_j : M_j \rightarrow C$  then we have that  $v$  is nonzero as a composition of epimorphisms and  $v \in \text{rad}^{n+1}(L, C)$  with  $L = M_j \in \mathbf{P}_j$  as required. By Proposition 5.6 in [4], either there exists  $p : N \rightarrow L$  such

that  $\pi_f = vp$  or there exists  $q : L \rightarrow N$  such that  $v = \pi_f q$ . In the first case, we get  $\pi_f \in \text{rad}^{n+1}(N, C)$  as  $v \in \text{rad}^{n+1}(L, C)$ , which is a contradiction. In the last case, as  $\text{tr}_{\mathbf{P}_k}(N) = \text{tr}_{\mathbf{P}_\infty}(N) = 0$  we have  $q = 0$  which implies  $v = 0$ , again a contradiction. Hence  $\text{Coker } f$  is postprojective.  $\square$

In [7], Coelho has considered the so-called  $\pi$ -components in  $\Gamma_A$ , which are components containing only postprojective modules. Hereditary algebras, or more generally, left glued algebras (see [1]), contain such components. These components can also be characterized for the fact that all its modules  $M$  satisfies  $\text{tr}_{\mathbf{P}_\infty}(M) = 0$  (see [7]).

**Corollary 3.2.** *Let  $\Gamma$  be a  $\pi$ -component and  $f : M \rightarrow N$  be an irreducible monomorphism in  $\Gamma$ . Then  $d_r(f) < \infty$  if and only if  $\text{Coker } f$  is postprojective.*

**Remark:** Although the above results may suggest that it is true that if  $f : M \rightarrow N$  is an irreducible monomorphism with  $N$  postprojective then  $d_r(f) < \infty$  if and only if  $\text{Coker } f \in \mathbf{P}$ , we alert that it is not the case. In Example 2.2(b) we see a source map  $f : \overline{S[1]} \rightarrow \overline{S[2]}$  which is a monomorphism of right degree equal to 1. We have that the  $A[S]$ -module  $\overline{S[2]}$  is postprojective but  $N[1] = \text{Coker } f \in \mathbf{P}_\infty$ .

Now we turn our attention to the case  $d_r(f) = \infty$ . We start with a definition.

**Definition 3.1.** Let  $M$  be an indecomposable in  $\mathbf{P}_\infty$ . We say  $M$  is  $\mathbf{P}_\infty$ -simple if every nontrivial submodule of  $M$  is postprojective.

**Remark:** We know that there is no bound on the lengths of the modules lying in any given infinite set of indecomposable nonisomorphic postprojective modules (see [3]). Hence, any  $A$ -module  $M$  must have a finite number of nonisomorphic postprojective submodules (if any). Therefore if  $M$  is  $\mathbf{P}_\infty$ -simple then there exists  $0 \leq n < \infty$  such that every nontrivial submodule of  $M$  is in  $\text{add } \mathbf{P}^n$ .

**Proposition 3.3.** *Let  $M$  be an indecomposable  $A$ -module in a regular component of  $\Gamma_A$ . If  $M$  is  $\mathbf{P}_\infty$ -simple then  $M$  is simple regular.*

*Proof.* If  $M$  is  $\mathbf{P}_\infty$ -simple then  $M$  must not have nontrivial regular submodules since all modules in regular components are in  $\mathbf{P}_\infty$ .  $\square$

Now we show that if  $f : M \rightarrow N$  is an irreducible monomorphism with  $\text{tr}_{\mathbf{P}_\infty}(N) = 0$  such that  $d_r(f) = \infty$  then  $\text{Coker } f$  is  $\mathbf{P}_\infty$ -simple. By Theorem in [7], we know that every monomorphism in a  $\pi$ -component satisfies this condition.

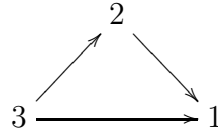
**Theorem 3.4.** *Let  $f : M \rightarrow N$  be an irreducible monomorphism with  $\text{tr}_{\mathbf{P}_\infty}(N) = 0$ . If  $d_r(f) = \infty$  then  $\text{Coker } f$  is  $\mathbf{P}_\infty$ -simple.*

*Proof.* We know by Theorem 3.1 that  $\text{Coker } f \in \mathbf{P}_\infty$ . Suppose by contradiction that  $\text{Coker } f$  is not  $\mathbf{P}_\infty$ -simple. Then there exists a nontrivial submodule  $X$  of  $\text{Coker } f$  such that  $X \in \mathbf{P}_\infty$ . Hence  $\text{Hom}(X, N) = 0$  since  $\text{tr}_{\mathbf{P}_\infty}(N) = 0$ . Consider the exact sequence  $0 \rightarrow M \xrightarrow{f} N \xrightarrow{g} \text{Coker } f \rightarrow 0$  and the inclusion morphism  $v : X \hookrightarrow \text{Coker } f$ . Then by Proposition 5.6 in [4] either there exists  $q : X \rightarrow N$  such that  $v = gq$  or there exists  $p : N \rightarrow X$  such that  $g = vp$ . The former leads to a contradiction because  $\text{Hom}(X, N) = 0$  implies  $q = 0$  which in turn implies  $v = 0$  and  $X = 0$ . The latter also leads to a contradiction for  $g$  being an epimorphism implies that  $v$  is an epimorphism and hence  $X = \text{Coker } f$  which is not possible. Therefore  $\text{Coker } f$  is  $\mathbf{P}_\infty$ -simple.  $\square$

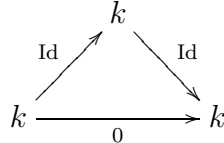
**Corollary 3.5.** *If  $f : M \rightarrow N$  is an irreducible monomorphism in a  $\pi$ -component such that  $\text{Coker } f$  lies in a regular component then  $\text{Coker } f$  is simple regular.*

We give now an example which illustrates the above corollary.

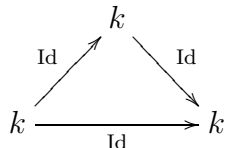
**Example 3.1.** Let  $A$  be a path algebra defined by the quiver:



For each vertex  $x$  in the quiver we set  $P_x$  the correspondent projective. Then the irreducible monomorphism  $f : P_1 \rightarrow P_3$  has infinite right degree since  $\text{Coker } f$  is not in the postprojective component.  $\text{Coker } f$  is as follows:



On the other hand, let  $\mu \in \text{rad}^2(P_1, P_3)$  be the composition of the irreducibles  $P_1 \rightarrow P_2 \rightarrow P_3$ . Then  $f' = f + \mu$  is also irreducible and  $\text{Coker } f'$  is as follows:



Since  $f$  has infinite right degree we have that  $f'$  has also infinite right degree. One can easily verify that  $\text{Coker} f$  and  $\text{Coker} f'$  lie in two different homogeneous tubes. By the above corollary both modules should be simple regular modules. And in fact that is what happens since both  $\text{Coker} f$  and  $\text{Coker} f'$  lie in the mouth of its respective tubes.

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